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A New Method to Study the Hawking Radiation from the Kerr-NUT Black Hole

Hong-Lin Liu · Zi-Xiang Liu · Jian-Song Hou · Shu-Zheng Yang

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Abstract Developing Hamilton-Jacobi method, we discuss the Hawking radiation of Kerr-NUT black hole by considering the self-gravitation interaction as well as the energy conservation and angular momentum conservation. The result shows that the factual spectrum deviates from the precisely thermal one and the tunneling rate is related to the change of Bekenstein-Hawking entropy, which is accordant with that obtained by Parikh and Wilczek's method and gives an interesting correction to the Hawking radiation of the black hole.

Keywords Kerr-NUT black hole · Energy conservation · Tunneling rate

1 Introduction

Over thirty years ago, Hawking proved thermal radiation of black hole [1, 2]. Since then, the thermodynamic properties of static, stationary and non-stationary black hole have been extensively studied. However, there are two points in dispute for years, one is the problem of the information lost and other is the cause of the mechanism concerned to the tunneling potential barrier. In addition, although Hawking radiation is treated as the quantum tunneling process, the quantum tunneling language has not been adopted to discuss the Hawking thermal radiation in the concerned documents. So, strictly speaking, the true quantum tunneling method wasn't adopted.

In 2000, Parikh and Wilczek applied the semi-classical quantum tunneling model to research on the Hawking radiation of the static Schwarzschild and Reissner-Nordström black holes [3, 4]. The derived result shows that the true radiation spectrum is not strictly thermal, but satisfies the underlying unitary theory under the consideration of energy conservation and self-gravitation interaction. Thus, the paradox of information lost receives a possible explanation. Meanwhile, they definitely pointed out the tunneling potential hill is produced by the self-gravitation interaction and the location of the horizons before and after the particle emission can be regarded as the two turning points of the tunneling potential hill. Following that, people studied the Hawking radiation of various space-times, such as Hemming

H.-L. Liu (✉) · Z.-X. Liu · J.-S. Hou · S.-Z. Yang
Institute of Theoretical Physics, China West Normal University, Nanchong 637002, China
e-mail: honglinliu@126.com

and Keski-Vakkuri to the Anti-de Sitter space-time [5], Medved to the de Sitter space-time [6] and Zhang et al. to the rotating space-times [7–9]. In 2005, Zhang et al. extended the work and further discussed the case of charged and massive particle [10–15]. All the results supported Parikh's opinion and gave a correction to the Hawking pure thermal spectrum.

In the same year, a different method was introduced to calculate the imaginary part of the action given by Angheben et al. [16–20]. The difference from Parikh's is mainly that such method concentrates on introducing the proper spatial distance and upon calculating the relativistic Hamilton-Jacobi equation. But the derived radiation spectrum only is the leader term due to the fact that the self-gravitation and energy conservation of the emitted particle is ignored. In this paper, we have investigated the Hawking radiation of the Kerr-NUT black hole by developing Hamilton-Jacobi method. Through considering self-gravitation interaction, energy conservation and angular momentum conservation, the radiation spectrum we obtained deviates from the precisely thermal one and the tunneling rate is connected with the change of the Bekenstein-Hawking entropy, which satisfies an underlying unitary theory and provides an interesting correction to Hawking pure thermal spectrum.

2 Dragging Coordinate Transformation

According to Ref. [21], the line element of the Kerr-NUT black hole is given by

$$ds^2 = -\frac{\Delta^2}{\rho^2}(dt - Pd\varphi)^2 + \frac{\sin^2\theta}{\rho^2}\left[(F + l^2)d\varphi - adt\right]^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2, \tag{1}$$

where

$$\begin{aligned} &F = r^2 + a^2, \quad \rho^2 = r^2 + (l + a\cos\theta)^2, \\ &\Delta^2 = r^2 - 2Mr + a^2 - l^2, \quad P = a^2 \sin^2\theta - 2l\cos\theta. \end{aligned} \tag{2}$$

The area and Bekenstein-Hawking entropy corresponding to the outer event horizon (r_+) of the black hole are expressed as

$$\begin{aligned} A_+ &= \int \sqrt{-g}d\theta d\varphi = 4\pi\left(r_+^2 + a^2 + l^2\right), \\ S_{\mathrm{BH}} &= (1/4)A_+ = \pi\left(r_+^2 + a^2 + l^2\right). \end{aligned} \tag{3}$$

According to $g_{00} = 0$ as well as the null super-surface equation $g^{\mu\nu}\partial_\mu f \partial_\nu f = 0$, the out infinite red-shift surface and the event horizon of the black hole can be obtained

$$r_{\mathrm{TLS}} = M + \sqrt{M^2 - a^2\cos^2\theta + l^2}, \quad r_+ = M + \sqrt{M^2 - a^2 + l^2}. \tag{4}$$

Obviously, the out infinite red-shift surface doesn't coincide with the event horizon, which is not convenient to study the Hawking radiation. And an effective approach to describe the Hawking radiation should be in the dragging coordinate system. Thus we perform the following dragging coordinate transformation

$$\dot{\varphi} = \frac{d\varphi}{dt} = -\frac{g_{03}}{g_{33}} \tag{5}$$

on the line element (1) and get

$$ds^2 = -\frac{\Delta^2\rho^2\sin^2\theta}{(r^2+a^2+l^2)\sin^2\theta - \Delta^2(a\sin^2\theta - 2l\cos\theta)^2}dt^2 + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2. \quad (6)$$

Now, the event horizon and out infinite red-shift surface are coincident with each other, which means geometry optics limit can be adopted here. Using WKB approximation, we can get the relationship between the tunneling rate and the imaginary part of radiation particle's action as $\Gamma \sim e^{-2\mathrm{Im}I}$.

3 The Hamilton-Jacobi Ansatz

The classical action I of the radiation particle satisfies the relativistic Hamilton-Jacobi equation as

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + u^2 = 0, \quad (7)$$

in which u is mass of the emitted particles, and $g^{\mu\nu}$ are the inverse metric tensor obtained from (6), namely

$$\hat{g}^{00} = -\frac{(r^2+a^2+l^2)\sin^2\theta - \Delta^2(a\sin^2\theta - 2l\cos\theta)^2}{\Delta^2\rho^2\sin^2\theta}, \quad g^{11} = \frac{\Delta^2}{\rho^2}, \quad g^{22} = \frac{1}{\rho^2}, \quad (8)$$

and others are null. Substituting them into (7) yields

$$-\frac{1}{P(r,\theta)}(\partial_t I)^2 + M(r,\theta)(\partial_r I)^2 + C(r,\theta)(\partial_\theta I)^2 + u^2 = 0, \quad (9)$$

where

$$P(r,\theta) = \frac{(r^2+a^2+l^2)\sin^2\theta - \Delta^2(a\sin^2\theta - 2l\cos\theta)^2}{\Delta^2\rho^2\sin^2\theta},$$

$$M(r,\theta) = \frac{\Delta^2}{\rho^2}, \quad C(r,\theta) = \frac{1}{\rho^2}. \quad (10)$$

Considering the axial symmetry of the black hole space-time, we carry out the separation variable to (9) as

$$I = -\omega t + W(r,\theta) + j\varphi, \quad (11)$$

where ω the energy of the emitted particle, $W(r,\theta)$ the generalized momentum and j the angular momentum with respect to the φ-axis. By substituting (11) into (9), we can get

$$\frac{\partial W(r,\theta)}{\partial r} = \frac{1}{\sqrt{P(r,\theta)M(r,\theta)}}\sqrt{(\omega - j\Omega)^2 - P(r,\theta)[C(r,\theta)[\partial_\theta W(r,\theta)]^2 + u^2]}, \quad (12)$$

where $\Omega = \frac{d\varphi}{dt}$. From above equation we can learn that the imaginary part of the emitted particle's action is only produced from the pole at the event horizon. According to the Ref. [16], for getting the correct result, the proper spatial distance should be introduced, which is defined by

$$d\sigma^2 = \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2. \quad (13)$$

In the paper, the particle is treated as an ellipsoid shell to tunnel across the event horizon and should not have motion in θ-direction ($d\theta = 0$). Namely

$$\sigma = \int [M(r,\theta)]^{-\frac{1}{2}} dr. \tag{14}$$

By applying the near-horizon approximation

$$P(r,\theta) = P'(r_+,\theta)(r - r_+) + \cdots, \qquad M(r,\theta) = M'(r_+,\theta)(r - r_+) + \cdots, \tag{15}$$

where $P'(r_+,\theta) = \frac{\partial P(r,\theta)}{\partial r}|_{r=r_+}$, $M'(r_+,\theta) = \frac{\partial M(r,\theta)}{\partial r}|_{r=r_+}$, we can obtain

$$\sigma = \frac{2}{\sqrt{M'(r_+,\theta)}} \sqrt{r - r_+} + \cdots. \tag{16}$$

From (12–16), we can derive

$$W(\sigma) = \frac{1}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} \int \frac{d\sigma}{\sigma} \sqrt{(\omega - j\Omega_+)^2 - P(r,\theta)[C(r,\theta)[\partial_\theta W(r,\theta)]^2 + u^2]}, \tag{17}$$

where $\Omega_+ = \frac{a}{r_+^2 + a^2 + l^2}$ is the angular velocity at the event horizon, and the solution is singular at $\sigma = 0$ which corresponds to the event horizon. Finishing the integral and substituting the result into (11) yields the imaginary part of the action as

$$\mathrm{Im} I = \frac{2\pi i}{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}} \left(\omega - \frac{ja}{r_+^2 + a^2 + l^2}\right). \tag{18}$$

And the temperature over the surface of the black hole is

$$T = \frac{\sqrt{P'(r_+,\theta)M'(r_+,\theta)}}{4\pi} = \frac{1}{2\pi} \frac{r_+ - M}{r_+^2 + a^2 + l^2} \tag{19}$$

this is consistent with the results derived from other methods. By using WKB approximation, the tunneling rate and the radiation spectrum of the emitted particle can be obtained. However, the result isn't accordant with recent research, namely its radiation spectrum is pure thermal. The reason is that the self-gravitation interaction wasn't taken into account. Now, let's incorporate it as well as the conservation of energy and angular momentum and move on discussing the Hawking radiation of the black hole.

4 The Tunneling Probability

We fix the ADM mass and angular momentum of the total space-time and allow these of the black hole to vary, when a particle with energy ω and angular momentum j tunnels out, the mass and angular momentum of the black hole should be changed into $M - \omega$ and $J - j$. Thus the imaginary part of the actual action should be

$$\begin{aligned} \mathrm{Im} I &= \pi \int_{(0,0)}^{(\omega,j)} \frac{r_+'^2 + a^2 + l^2}{r_+' - (M - \omega')} \left(d\omega' - \frac{a}{r_+'^2 + a^2 + l^2} dj'\right) \\ &= -\pi \int_{(M,J)}^{(M-\omega,J-j)} \frac{r_+'^2 + a^2 + l^2}{r_+' - (M - \omega')} \left[d(M - \omega') - \frac{a}{r_+'^2 + a^2 + l^2} d(J - j')\right], \end{aligned} \tag{20}$$

where

$$J - j' = (M - \omega')a, \quad r'_+ = (M - \omega') + \sqrt{(M - \omega')^2 - a^2 + l^2}. \tag{21}$$

Substituting (21) into (20), and finishing the integral, the imaginary part of can be obtained

$$\begin{aligned}
\mathrm{Im} I &= -\pi \int_{(M)}^{(M-\omega)} \frac{r'^2_+ + a^2 + l^2}{\sqrt{(M - \omega')^2 - a^2 + l^2}} d(M - \omega') \\
&= -\pi \left[(M - \omega)^2 - M^2 + (M - \omega)\sqrt{(M - \omega)^2 - a^2 + l^2} - M\sqrt{M^2 - a^2 + l^2} \right] \\
&= -\frac{1}{2} \Delta S_{\mathrm{BH}},
\end{aligned} \tag{22}$$

where $\Delta S_{\mathrm{BH}} = S_{\mathrm{BH}}(M - \omega) - S_{\mathrm{BH}}(M)$ is the change of Bekenstein-Hawking entropy of the Kerr-NUT black hole [22]. Therefore the tunneling rate is

$$\Gamma \sim \mathrm{e}^{-2\mathrm{Im} I} = \mathrm{e}^{\Delta S_{\mathrm{BH}}}. \tag{23}$$

Obviously, the radiation spectrum given by (23) is not pure thermal, which gives a correction to the Hawking radiation of the black hole.

Now let's expand the (23) in $\omega - \omega_0$, we get

$$\Gamma \sim \mathrm{e}^{\Delta S_{\mathrm{BH}}} = \mathrm{e}^{-\frac{\omega-\omega_0}{T}[1-(\omega-\omega_0)\frac{r_+^2+a^2+l^2}{r_+^4}(M+\sqrt{M^2-a^2+l^2}-\frac{M(a^2-l^2)}{2(M^2-a^2+l^2)})+\cdots]} \tag{24}$$

when neglecting high-order term of $\omega - \omega_0$, the Hawking pure thermal spectrum can be obtained. We therefore come to the conclusion that the actual radiation spectrum of Kerr-NUT black hole is not precisely thermal, which provides an interesting correction to Hawking pure thermal spectrum. When $l = 0$, the Kerr-NUT black hole reduced to Kerr black hole. Therefore, we can get the tunneling rate of Kerr black hole. And substituting $a = l = 0$ into (24), the tunneling rate of Schwarzschild black hole can be also obtained.

In this paper, we have discussed the Hawking radiation of Kerr-NUT black hole by developing Hamilton-Jacobi method. The radiation spectrum we obtained is more accurate and provides an interesting correction to Hawking pure thermal spectrum.

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